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Regular p-Groups and Groups of Maximal Class

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INTRODUCTION

In this note we consider only finite p-groups. Such a group is *of maximal class*, if it has order p^n and class $n - 1$. The theory of such groups was developed in detail by Blackburn [2; or 3, III.14]. Our purpose here is to provide alternative proofs for some of Blackburn's main results. It turns out that, by applying the theory of regular p-groups [3, III.10], one can avoid some of the calculations involved in the original proofs. A simple lemma which is needed for that (Lemma 1) turns out to be also useful in simplifying the proof of a result of Brisley and MacDonald [1, 3.1], which characterizes regular metabelian p-groups, and we start with this result.

The notation and terminology, which are standard, follow [3].

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LEMMA 1. *Let G be a regular p-group of class p, in which G' has exponent p. Then G satisfies the identity*

$$[x, y, y, \dots, y] = 1 \tag{1}$$

where y appears $p - 1$ times.

Proof. This is proved in [3, III.9.7] for groups of exponent p. However, the proof there uses only the identity $(xy)^p = x^p y^p$, which holds in a regular p-group in which G' has exponent p.

LEMMA 2. *Let G be a metabelian nilpotent group of class c. Then G satisfies*

(a) *If $a, b \in G$ and $u \in G'$, then*

$$[u, a, b] = [u, b, a]. \tag{2}$$

(b) *If* $a_1, \ldots, a_{n-1}, a_{n+1}, \ldots, a_c, b_n, c_n \in G$,

$$[a_1, \ldots, a_{n-1}, b_n c_n, a_{n+1}, \ldots, a_c]$$
$$= [a_1, \ldots, a_{n-1}, b_n, a_{n+1}, \ldots, a_c][a_1, \ldots, a_{n-1}, c_n, a_{n+1}, \ldots, a_c] \quad (3)$$

Proof. Since G' is abelian, G induces on G' an abelian group of automorphisms which spans a commutative subring of the ring of endomorphisms of G'. This subring contains the maps $\alpha = a - 1$ for which $u^\alpha = [u, a]$ and $\beta = b - 1$ $(u \in G')$, hence

$$[u, a, b] = u(\alpha\beta) = u(\beta\alpha) = [u, b, a].$$

For part b, denote $u = [a_1, \ldots, a_{n-1}]$, then

$$[u, b_n c_n] = [u, b_n][u, c_n][u, b_n, c_n]$$

(if $n = 1$, we apply similarly the identity

$$[b_1 c_1, a_2] = [b_1, a_2][c_1, a_2][b_1, a_2, c_1]).$$

Since commutation is an endomorphism in G', we find that

$$[u, b_n c_n, a_{n+1}, \ldots, a_c]$$
$$= [u, b_n, a_{n+1}, \ldots, a_c][u, c_n, a_{n+1}, \ldots, a_c][u, b_n, c_n, a_{n+1}, \ldots, a_c],$$

and the last factor is 1, being of weight $c + 1$.

We now come to the Brisley–MacDonald result.

THEOREM 3. *A metabelian p-group G is regular, if and only if each 2-generator subgroup H of G satisfies* $H_p \subseteq \mho_1(H_2)$.

Proof. That groups satisfying the stated condition are regular, is both well-known and trivial. Assume, then, that $H = \langle a, b \rangle$ is 2-generator metabelian regular p-group. We may assume $\mho_1(H_2) = 1$. If $H_p \neq 1$, then $H_p \neq H_{p+1}$, so we may assume $H_{p+1} = 1$ and try to prove $H_p = 1$.

Denote $d_i = [a, b, \ldots, b, a, \ldots, a]$, where b appears i times and a appears $p - i$ times. By Eq. (2), each commutator of weight p in a and b is one of the d_i's, so the d_i's generate H_p. Substitute $x = a$, $y = ab^n$ in (1), and apply (3) and (2) to express the resulting commutator as a product of powers of the d_i's. Then d_i appears with exponent $\binom{p-2}{i-1} n^i$. We can regard H_p, written additively, as a vector space over $GF(p)$, and then our substitutions in (1), for $n = 1, 2, \ldots, p-1$, yield a system of $p-1$ homogeneous equations in $d_1, \ldots, d_{p-1}$, with matrix $(a_{in}) = (\binom{p-2}{i-1} n^i)$, having determinant

$$1 \cdot (p-2) \cdots \binom{p-2}{p-2} \cdot (p-1)!\, \Pi(n-m) \not\equiv 0(p)$$

and so each $d_i = 1$, and $H_p = 1$.

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We now pass to groups of maximal class. Throughout this section, let G be such a group of order p^n. Then $|G : G_2| = p^2$ and $|G_i : G_{i+1}| = p$ for $i = 2, \ldots, n-1$. If N is a normal subgroup of G of index p^i $(i \geqslant 2)$, then $N \supseteq G_i$, so $N = G_i$ by equality of indices. Also, for any $i = 2, \ldots, n-2$, G_i/G_{i+2} is a noncentral normal subgroup of order p^2 of G/G_{i+2}, hence its centralizer is a maximal subgroup, i.e., $C_G(G_i/G_{i+2})$ is a maximal subgroup of G.

Lemma 4 (cf. 3, III.14.14 and 14.21). *If $|G| \leqslant p^{p+1}$ then $|\mho_1(G)| \leqslant p$ and G' has exponent p. If $|G| = p^{p+1}$ then G is irregular and $|\mho_1(G)| = p$.*

Proof. G involves $n-3$ factors G_i/G_{i+2}, for $i = 2, \ldots, n-2$. Here $n - 3 \leqslant p - 2$, so since G contains $p+1$ maximal subgroups, we can find two maximal subgroups of G which are different from all the subgroups $C_G(G_i/G_{i+2})$. Thus we can find also two generators, a and b, for G, none of which centralizes any factor G_i/G_{i+2}. Suppose that $a^p \in G_i$, but $a^p \notin G_{i+1}$, for some $i \leqslant n-2$. Then $G_i = \langle a^p, G_{i+1} \rangle$, so that a centralizes G_i/G_{i+2}, a contradiction. Therefore, $a^p \in G_{n-1}$, and similarly $b^p \in G_{n-1}$. The group G/G_{n-1}, of order p^p at most, is regular and generated by two elements of order p, hence it has exponent p, i.e., $\mho_1(G) \subseteq G_{n-1}$.

Let M be any maximal subgroup of G. Then M is regular and $|\mho_1(M)| \leqslant p$, so $|M : \Omega_1(M)| \leqslant p$, and $\Omega_1(M)$ has index at most p^2 in G. As G' is the only normal subgroup of index p^2 of G, it follows that $G' \subseteq \Omega_1(M)$ and G' has exponent p.

Let $c_i = [a, b, \ldots, b]$, where b appears $i-1$ times. It is well known that $G_2 = \langle c_2, G_3 \rangle$. Assume we have already proved that $G_i = \langle c_i, G_{i+1} \rangle$. Then $c_{i+1} = [c_i, b] \notin G_{i+2}$, otherwise b would centralize G_i/G_{i+2}, and this implies that $G_{i+1} = \langle c_{i+1}, G_{i+2} \rangle$.

Now let $|G| = p^{p+1}$. Then $G_p = \langle c_p \rangle \neq 1$, hence $c_p \neq 1$. By Lemma 1, G is irregular. Then $\mho_1(G) \neq 1$, so that $|\mho_1(G)| = p$.

Theorem 5. *Let G be a group of maximal class of order p^n, and let $n \geqslant$*

$p + 2$. *Then G contains a unique regular maximal subgroup G_1, and all other maximal subgroups of G are of maximal class. Moreover, for $i, j = 1, 2, \ldots, n-1$ we have* $[G_i, G_j] \subseteq G_{i+j+1}$ (cf. 3, III.14.6(b) and 14.22).

Proof. Applying Lemma 4 to G/G_{p+1}, we obtain $\mho_1(G)\, G_{p+1} = G_p$. Since one of $\mho_1(G)$, G_{p+1} contains the other, this means that $\mho_1(G) = G_p$. Now we obviously have

$$\mho_1(G) = \langle \mho_1(M) \mid M \text{ maximal in } G \rangle$$

and so we cannot have $\mho_1(M) \subseteq G_{p+1}$ for all such M. Choose a maximal subgroup G_1 such that $\mho_1(G_1) \not\subseteq G_{p+1}$, then $\mho_1(G_1) = G_p$. But then $| G_1 : \mho_1(G_1)| \leqslant p^{p-1}$, and G_1 is regular by [3, III.10.13].

By properties of regular groups, we obtain $| \Omega_1(G_1)| = p^{p-1}$, and so $\Omega_1(G_1) = G_{n-p+1}$. For any $i = 1, 2, \ldots, n-p+1$ we have $G_{n-p+1} \subseteq \Omega_1(G_i) \subseteq \Omega_1(G_1)$, and so $G_{n-p+1} = \Omega_1(G_i)$. As G_i is regular, we have also $| G_i : \mho_1(G_i)| = p^{p-1}$, so that

$$\mho_1(G_i) = G_{i+p-1}, \qquad i = 1, 2, \ldots, n-p+1 \tag{4}$$

Assume for the moment that $n = p + 2$. Let M be any regular maximal subgroup of G. If $\mho_1(M) \subseteq G_{p+1}$, then $| \Omega_1(M)| = | M : \mho_1(M)| \geqslant p^p$, hence $\Omega_1(M) \supseteq G_2$ (the unique normal subgroup of order p^p), and so $| \Omega_1(G_1)| \geqslant | G_2 | = p^p$, a contradiction. Thus $\mho_1(M) = G_p$. From $| G_3 | = | \Omega_1(G_1)| = p^{p-1}$ we see that $G_3 = \Omega_1(G_1)$ has exponent p and so, by [3, III.10.8(a)].

$$[M, G_p] = [M, \mho_1(M)] = \mho_1([M, M]) \subseteq \mho_1(G_3) = 1, \qquad M = C_G(G_p).$$

This makes M unique, and $M = G_1$.

Let M be a maximal subgroup $\neq G_1$. There exists a central series

$$M \supsetneq G_3 \supsetneq G_4 \supsetneq \cdots \supsetneq G_{p+2} = 1 \tag{5}$$

of M, in which the first factor has order p^2, and the others—order p. Since M is irregular and of order p^{p+1}, we must have cl $M = p$, M is of maximal class, and (5) is the lower central series of M. It follows that M does not centralize any factor G_i/G_{i+2} for $i \geqslant 3$ and M also does not centralize G_2/G_4, or M/G_4 would be abelian. However, $C_G(G_i/G_{i+2})$ is a maximal subgroup of G, and so equals G_1, i.e.,

$$[G_1, G_i] \subseteq G_{i+2} \tag{6}$$

Returning to the general case $n \geqslant p + 2$, we now prove (6) by induction on i. First, for $i \leqslant p$, (6) follows by considering G/G_{p+2}. Let $i > p$, then, employing (4), properties of regular groups, and the induction assumption, we get

$$[G_1, G_i] = [G_1, \mho_1(G_{i-p+1})] = \mho_1([G_1, G_{i-p+1}]) \subseteq \mho_1(G_{i-p+3}) = G_{i+2}.$$

Next we establish, by induction on i,

$$[G_i, G_j] \subseteq G_{i+j+1}. \tag{7}$$

This has already been proven for $i = 1$. Assuming (7) for a given i, and *all* j, we evaluate

$$\begin{aligned}[G_{i+1}, G_j] &= [G_i, G, G_j] \subseteq [G_j, G_i, G][G, G_j, G_i] \\ &\subseteq [G_{i+j+1}, G][G_{j+1}, G_i] = G_{i+j+2}\end{aligned}$$

(notice that it is true that $G_2 = [G_1, G]$).

Finally, since $G_1 = C_G(G_i/G_{i+2})$, we see that $[M, G_i] \nsubseteq G_{i+2}$, for any maximal subgroup $M \neq G_1$, which forces $[M, G_i] = G_{i+1}$, for $i = 2,\dots, n-2$, and so $M \supseteq G_3 \supseteq \cdots \supseteq G_n = 1$ is the lower central series of M, so $\mathrm{cl}\, M = n - 2$ and M is of maximal class.

The following result is a special case of [2, 3.9].

THEOREM 6. *Let G be a finite p-group. If G/G_{p+1} is of maximal class, so is G.*

Proof. Let G be of class n, where $n > p$. Let Z be a minimal normal subgroup of G contained in G_n. By induction, G/Z is of maximal class. If $Z = G_n$, then G itself is of maximal class, so we assume $Z \neq G_n$. Then $|G_n/Z| = p$, and $|G_n| = p^2$. Working (mod Z), we see that $Z(G) \subseteq G_n$, hence $Z(G) = G_n$. Let $N \neq 1$ be any normal subgroup of G, and let Z be a minimal normal subgroup of G contained in N. Working in G/Z, we see that N is a maximal subgroup, $N = G_i$ for some i, or $N \subseteq Z(G)$.

Take $N = \mho_1(G)$, then Lemma 4 implies $NG_{p+1} = G_p$, and since $Z(G) = G_n \subseteq G_{p+1}$, we find that $N = G_p$. We can now repeat the first part of the proof of Theorem 5, to find in G a regular maximal subgroup G_1, such that $\mho_1(G_1) = G_p$, and $|\Omega_1(G_1)| = p^{p-1}$. In particular, we find that $Z(G)$ is elementary (for $p > 2$; for $p = 2$ we see that G_1 is cyclic, and finish the proof by checking the list of groups with a cyclic maximal subgroup), and so we can find two distinct minimal normal subgroups of G, say Z_1 and Z_2. Then G/Z_i is of maximal class, hence $G_{n-p+1}/Z_i = \Omega_1(G_1/Z_i)$ is of exponent p, and G_{n-p+1} itself, of order p^p, is of exponent p, contradicting $|\Omega_1(G_1)| = p^{p-1}$.

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